THE SULLIVAN CONJECTURE

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These are notes on Lannes' proof [Lan92] of the Sullivan conjecture (originally proved by Miller [Mil84]). These notes were written for a pair of talks as part of Elden Elmanto's Kan Seminar. These notes are much shortened and slightly modernized from the lecture notes by Lurie [Lur07].

Contents

1. Introduction

Theorem 1.1 (The Sullivan Conjecture)**.** *Suppose that X is a based finite space and G is a finite group, then* $\mathrm{Hom}(BG, X) \simeq *$.

We can rephrase this by saying that that map $X \to X^{hG}$ is a homotopy equivalence where X is given the trivial G action.

To simplify our proof, our space X will always be simply connected, but we can also easy deal with nilpotent spaces. The original proof doesn't require any assumption on the fundamental group, but this proof is much less conceptual.

1.1. Motivation from Algebraic Geometry.

Suppose that X is an algebraic variety over $\mathbb{R}$. We might try to understand the topology of the space corresponding to the real points $X(\mathbb{R})$ of X. We know that $X(\mathbb{R}) = X(\mathbb{C})^{\mathbb{Z}/2}$. There are methods to recover the p-aidic completion of $X(\mathbb{C})^\vee_p$ in purely algebraic terms. We get a homotopy action of $\mathbb{Z}/2$ on each $X(\mathbb{C})^\vee_p$ (but *not* a genuine action), so it is reasonable to ask how close $(X(\mathbb{C})^\vee_p)^{h\mathbb{Z}/2}$ is to the space $X(\mathbb{R}) = X(\mathbb{C})^{\mathbb{Z}/2}$. Take for example $\mathbb{P}^1$. When $p \neq 2$ this is a terrible approximation, $(\mathbb{P}^1(\mathbb{C})^\vee_p)^{h\mathbb{Z}/2}$ is simply connected, far from the circle (whose only defining feature is *not* being simply connected). When $p = 2$ this turns out to be an isomorphism on $\mathbb{Z}/2$-cohomology.

This turns out to be the consequence of a variation of the conjecture above.

Theorem 1.2 (The Actual Sullivan Conjecture). *Suppose that X is a simply connected finite G-space for G a finite p group. Then*

$$X^G \to (X_p^\vee)^{hG}$$

induces an isomorphism on $\mathbb{Z}/p$-cohomology.

This turns out to be true and can be proven using the same set of tools as our main theorem (although we won't discuss this proof). Our main theorem is a degenerate version of this when the action of G is trivial, but it yields a stronger result.

In this talk we aim to give a proof of the Sullivan conjecture, and a review of the related homotopy theory. We postpone reviewing anything in favor of discussing the proof strategy.

1.2. **Strategic Outline.** Sullivan's conjecture is about the space of maps between two topological spaces. Our proof is a dance of decomposing each space in various ways. After that the real work begins.

Step 1. To start we fracture X as the pullback

$$\begin{array}{ccc} X & \longrightarrow & \prod_p \widehat{X}_p \\ \big\downarrow & \lrcorner & \big\downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p \widehat{X}_p)_{\mathbb{Q}} \end{array}$$

so we can then prove the theorem at each prime and rationally. As a reminder, $\widehat{X}_p$ is the p-aidic completion of X. Rationally we know that for any Y we get $\mathrm{Hom}(BG, Y_{\mathbb{Q}})$ is exactly $\mathrm{Hom}(BG_{\mathbb{Q}}, Y_{\mathbb{Q}})$ but then since G is finite $BG_{\mathbb{Q}} \simeq *$ (the homology of a finite group is always torsion).

Step 2. We now work one prime p at a time, and prove that we can replace G with a p-group.

Step 3. We now have reduced this to proving a p-profinite version of the Sullivan Conjecture.

Theorem 1.3 (The p-profinite Sullivan Conjecture). *For X a finite space and K a connected p-finite space the map $X_p^\vee \to (X_p^\vee)^K$ is a homotopy equivalence.*

We care about when K is an Eilenberg-Maclane space of a finite p-group. We reduce this profinite version of the Sullivan conjecture to the case where $K = K(\mathbb{Z}/p, 1)$.

Step 5. We now prove this limited version of the p-profinite Sullivan conjecture using Lannes' T-functor on the category of unstable modules over the Steenrod algebra. Showing that this functor has the desired properties is the heart of this proof.

2. Reductions through Profinite Homotopy Theory

2.1. p-adic Homotopy Theory.

Definition 2.1. A *p-finite space* is a space for which, each homotopy group is a finite p-group, there are finitely many path components, and the homotopy groups eventually vanish. These form the full subcategory of spaces $\mathcal{S}_p$ called p-finite spaces.

Definition 2.2. The category $\mathcal{S}_p^\vee$ of *p-profinite spaces* is the profinite completion of $\mathcal{S}_p$. Think of objects in this category as formal cofiltered limits of p-finite spaces, with morphisms given by partial maps of these filtered diagrams.

Definition 2.3. There is a natural map $\mathcal{S}_p^\vee \to \mathcal{S}$ given by taking a formal limit to the actual limit. This functor admits (for formal reasons) a left adjoint called the *p-profinite completion.* We denote the p-profinite completion by $X_p^\vee$.

Definition 2.4. $\widehat{X}_p$, the *p-adic completion* of X is given by mapping X under the composition

$$\mathcal{S} \to \mathcal{S}_p^\vee \to \mathcal{S}.$$

The image of this composition will be our p-completed spaces

We now reach the upshot of p-completed spaces.

Proposition 2.5. *We can check if a map $\widehat{X}_p \to \widehat{Y}_p$ is an equivlence by looking at $\mathbb{F}_p$ cohomology.*

2.2. **Sylow Subgroups.**

Proposition 2.6. *Assume the Sullivan Conjecture for $\widehat{X}_p$ when G is a p-group. Then the Sullivan conjecture is true for $\widehat{X}_p$ where G is any group.*

Proof. Let $H \leq G$ be a p-sylow subgroup. The idea of this proof is roughly that if H were normal we could take the quotient G/H and say that it's $\mathbb{F}_p$-cohomology vanishes. Then using the fact that H is a p-group we would be done using a two out of three property. Since we may not be able to take this quotient G/H in the category of groups, we have to work harder.

Define M_n to be the simplicial set $(G/H)^{n+1}$ with the G-action given by diagonal multiplication. Then we get that $M_\bullet$ is contractible so the simplicial space $K_\bullet = (M_\bullet)_{hG}$ is a model for BG. Now define $K'_n = \pi_0 K_n$, we get that $K'_\bullet$ is just the G fixed points of $M_\bullet$. Now elements of K'_n look like tuples $(g_0H, \dots, g_nH)$ with two sequences being equivalent if they are identified under the action of G.

Now the fibre of the map $K_n \to K'_n$ over $(g_0H, \dots, g_nH)$ is the space BP where P is subgroup given by

$$\bigcap_{i=0}^{n} g_i H g_i^{-1}.$$

The group P is always a p-group. Thus *using our assumption* we get that the map $\widehat{X}_p \to \widehat{X}_p^{BP}$ is an equivalence. Therefore to show that $\widehat{X}_p \to \widehat{X}_p^{K_\bullet} \simeq \widehat{X}_p^{BG}$ it is enough to show that $\widehat{X}_p \to \widehat{X}_p^{K'_\bullet}$ is a homotopy equivalence.

To show that $\widehat{X}_p \to \widehat{X}_p^{K'_\bullet}$ is a homotopy equivalence, it is enough to show that $H^\bullet(*, \mathbb{F}_p) \to H^\bullet(K_\bullet, \mathbb{F}_p)$ is an equivalence. For this we can contract the chain complex

$$\cdots \to \mathbb{F}_p[K_1] \to \mathbb{F}_p[K_0] \to \mathbb{F}_p[((G/H)^0)_G] \to 0.$$

This is given by the formula

$$(g_0H, \dots, g_nH) \mapsto \frac{1}{|G/H|} \sum_{gH \in G/H} (gH, g_0H, \dots, g_nH).$$

□

Because the functor $\operatorname{Hom}(BG, -)$ commutes with limits, we can reduce to proving this for $X_p^\vee$. We have now finished the reduction to the p-profinite case.

2.3. **Atomic Profinite Objects.**

Definition 2.7. A p-profinite space K is atomic if the functor $X \mapsto X^K$ preserves finite homotopy colimits.

Theorem 2.8. *Any connected p-finite space is atomic.*

Although we could prove this in full generality, we are satisfied with $K = K(G, 1)$ for a finite p-group G. The proof of the general case proceeds by induction on the Postnikov tower. It is also worth remarking that atomicity is a general categorical notion. Being atomic means that "mapping out of K behaves like mapping out of the point". This is codified by the following.

Proposition 2.9. *A p-profinite space is atomic if the projection map*

$$(\mathcal{S}_p^\vee)_{/K} \xrightarrow{\pi_{K*}} (\mathcal{S}_p^\vee)_{/*}$$

preserves finite colimits.

This reformulation will allow us to make another reduction. Using the fact that $(\mathcal{S}_p^\vee)_{[-]}$ defines a four-functor formalism for p-profinite spaces.

Proposition 2.10. *If*

$$\begin{array}{ccc} F & \xrightarrow{i} & E \\ {\scriptstyle \pi_F}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ * & \xrightarrow{b} & B \end{array}$$

is a fibre sequence and F and B are atomic, then so is E.

Proof. We want to show that $\pi_{E*} \simeq \pi_{B*} \circ p_*$ preserves finite colimits. π_{B*} does because B is a atomic, so it is enough to verify that p_* does. p_* preserving finite colimits can be checked pointwise (we are working in functor categories after all), so we just need to verify that $b^* p_*$ preserves finite colimts. Using base change we see that

$$b^* p_* \simeq \pi_{F*} i^*.$$

Both of the right hand functors preserve finite colimits because i^* is a left adjoint and F is atomic. □

Since we can build any finite p-group G from $\mathbb{Z}/p$ by extensions. We can show that $K(G, 1)$ is atomic by showing that $K(\mathbb{Z}/p, 1)$ is atomic. This is where the real work begins.

3. Lannes' T-Functor

3.1. **Proving The Sullivan Conjecture.**

Having reduced Sullivan's conjecture to a version for p-profinite spaces where our group is a finite dimensional vector space V over $\mathbb{F}_p$, we find ourselves in need of some new ideas. Our hammer will be the following.

Theorem 3.1 (Lannes)**.** *For any vector space V over $\mathbb{F}_p$ there is an endofunctor T_V on category of unstable modules $\mathcal{U}$ over the steenrod algebra $\mathcal{A}_p$ such that*

(1) T_V preserves finite colimits.
(2) For a p-profinite space X, there is a map $T_V H^*(X; \mathbb{F}_p) \simeq H^*(X^{BV}; \mathbb{F}_p)$.

We will have an in depth discussion of the Steenrod algebra and its category of unstable modules, and we will discuss where this functor T_V comes from. It is also worth remarking that T_V enjoys many other nice properties, such as monoidality, that we won't discuss. Before our discussion of the steenrod algebra and the T-functor, we will finish proving the Sullivan Conjecture.

Corollary 3.2. *V is atomic.*

Proof. T_V preserves pushouts. Since we can build any finite space using pushouts with a point, we have reduced this to the case when X is a point. Here, it is immediate. □

3.2. **The Steenrod Algebra.**

Remark 3.3. We limit our discussion to at the prime 2 but odd primes are not conceptually harder, except instead of being generated by Steenrod squares the algebra is generated by Bockstein operations and power operations. With this being said, all homology and cohomology is taken with $\mathbb{F}_2$ coefficients for the remainder of these notes.

Definition 3.4. The Steenrod algebra $\mathcal{A}$ is the $H\mathbb{F}_2$ (graded) algebra $H\mathbb{F}_2^*(H\mathbb{F}_2)$.

The algebra is often called the "cohomology of cohomology". Given a spectrum X, we calculate its cohomology as $\underline{\mathrm{Hom}}(X, H\mathbb{F}_2)$. This becomes a module over $\mathcal{A}$ using post composition.

The elements of this algebra are cohomology operations, in the sense that the given maps $H\mathbb{F}_2^* X \to H\mathbb{F}_2$ which play nicely with the properties of cohomology. We have a full understanding of the structure of $\mathcal{A}$.

Theorem 3.5 (Serre-Cartan, Ádem)**.** *The algebra $\mathcal{A}$ is the quotient of the algebra $\mathbb{F}_2[\mathrm{Sq}^i : i \in \mathbb{Z}]$ with $\deg(\mathrm{Sq}^i) = i$ where we impose the so called Ádem relations*

$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k$$

and we impose that $\mathrm{Sq}^0 = 1$.

Now suppose that X is a space. Then as remarked above $H^*(X)$ is a module over the Steenrod algebra. However, not all modules over the Steenrod algebra are given as the cohomology of some space, for example the module $H^*(X)$ always satisfies the following condition.

Definition 3.6. A module M over $\mathcal{A}$ is *unstable* if $\mathrm{Sq}^k(x) = 0$ for all x homogenous of degree less than k. We denote the category of unstable modules $\mathcal{U}$.

The category of unstable modules falls into the "wrong category, right properties paradigm". Not all unstable modules come from the cohomology of spaces, but the category of unstable modules has good properties.

3.3. **The Functor T_V.**

We are now in a position to understand the theorem which claims the existence of T_V. We move on the building such a functor.

To begin we remark that the category $\mathcal{U}$ is endowed with a tensor product coming from the one on all modules. One can think of this as coming from the product of spaces, but not all unstable modules are the cohomology of some space.

Definition 3.7. Suppose that $M \in \mathcal{U}$ is a *finite type module*, a module whose graded pieces are finitely generated. Then T_M is the functor which is left adjoint to $- \otimes M$.

Such a functor exists using the adjoint functor theorem, and this theorem also tells us that T_M preserves colimits. This is property (1) of the functor T_V.

From now forward we restrict to the case where M is $H^*(BV)$ for some finite dimensional vector space V, in this case we write T_V for $T_{H^*(BV)}$. Now we have an evaluation map

$$X^{BV} \otimes BV \to X$$

which determines a map

$$H^*(X) \to H^*(X^{BV}) \otimes H^*(BV)$$

which has an adjoint map

$$T_V H^*(X) \to H^*(X^{BV}). \tag{1}$$

It is exactly this map adjoint to evaluation which will turn out to be an equivalence, which is property (2) of the functor T_V.

Theorem 3.8. *The map adjoint to evaluation 1 is an equivalence for any p-finite space X.*

Proof Sketch. For this proof, we need another property of T_V which we didn't list earlier.

Proposition 3.9. *The functor T_V is monoidal.*

Because the cohomology functor takes cofiltered limits to colimits and the functor T_V is exact, we can check this on p-finite spaces. To do this we will induct on a refined p-finite Posnikov tower. Here each of the fibres are a $K(\mathbb{F}_p, n)$. So we need to prove this theorem for the Eilenberg Maclane spaces $K(\mathbb{F}_p, n)$, and we also need to show that the property of the map 1 being an equivalence is closed under pullbacks.

For pullbacks,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

we will get a pushout in the category of E_∞ algebras (over $\mathbb{F}_2$)

$$C(X') \simeq C(Y') \otimes_{C(Y)} C(X).$$

Checking that we actually get such a pushout takes a little work. Lurie calls this "convergence of the Eilenberg-Moore spectral sequence". Let i_X denote the map from X to Y. Then we can view cohomology of X as cohomology of the local system $i_{X*}\underline{\mathbb{F}_2}_X$. Similarly for X' and Y'. Now because we have a pushout, $i_{X*}\underline{\mathbb{F}_2}_X$, now we can check the equivalence of local systems on Y

$$i_{X*}\underline{\mathbb{F}_2}_X \otimes i_{Y'*}\underline{\mathbb{F}_2}_{Y'} \simeq i_{X'*}\underline{\mathbb{F}_2}_{X'}$$

pointwise using the Künneth theorem, as long as Y is finite. With this in hand, the equivlence $C(X') \simeq C(Y') \otimes_{C(Y)} C(X)$ gives the spectral sequence

$$E_1^{-*,q} = H^*(Y') \otimes \underbrace{H^*(Y) \otimes \cdots \otimes H^*(Y)}_{q \text{ times}} \otimes H(X) \Rightarrow H^*(X)$$

in the category $\mathcal{U}$. To say this, one must convince themself that $\mathcal{U}$ is closed under subquoitents as a subcategory of all modules over the $\mathcal{A}$. Now the functor $(-)^{BV}$ preserves finite objects, so we have another spectral sequence ${E'}_1^{-*,q}$ for the pushout

$$\begin{array}{ccc} X'^{BV} & \longrightarrow & X^{BV} \\ \downarrow & \lrcorner & \downarrow \\ Y'^{BV} & \longrightarrow & Y^{BV} \end{array}.$$

Then the adjoint map and the fact that T_V preserves colimits gives a map of spectral sequences $T_V E_1^{-*,q} \to {E'}_1^{-*,q}$. The fact that T_V is monoidal and our hypothesis gives that we get an isomorphism on spectral sequences, and since T_V is exact we get that the corresponding map on $T_V H^*(X) \to H^*(X^{BV})$ is an isomorphism.

For Eilenberg Maclane spaces, the module $H^*(K(\mathbb{F}_2, n))$ will be free on one generator in degree n which we call $F(n)$, then computes that $T_V(F(n)) \simeq F(n) \otimes \cdots \otimes F(0)$. Once this this is computed, the computation then follows because $H^*(\mathrm{Hom}(\mathbb{F}_2, K(\mathbb{F}_2, n)))$ is exactly $F(n) \otimes \cdots \otimes F(0)$ (in fact the corresponding fact is true at the level of spaces). To see that $T_V(F(n)) \simeq F(n) \otimes \cdots \otimes F(0)$, one check thats T_V viewed as a functor on the category $\mathcal{K}$ of *algebras* in $\mathcal{U}$, T_V is still left adjoint to $- \otimes H^*(BV)$, and this allows us to compute the mapping space.

□

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