AN ASSEMBLY THEORETIC PROOF OF DWW

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These are notes for Alexander Kupers' Simple Homotopy Week. They follow very closely the paper of Raptis-Steimle [RS20]. This follows a talk discussing cobordism categories and introducing the Dwyer-Wiess-Williams theorem.

Contents

References

[RS20] G. Raptis and W. Steimle, *Topological manifold bundles and the a-theory assembly map*, 2020, `arXiv:1905.01868`. 1

1. Topological Cobordism Categories

We want to work with not only the smooth manifold bundles, but also topological manifold bundles

$$E \xrightarrow{\pi} B.$$

To begin we will talk about topological cobordism categories.

1.1. **Tangent Microbundles.** To begin with, E does not have a tangent bundle because it is not smooth, but it does have a *tangent micro-bundle*. If E were smooth we could view TE inside of $E \times E$ as follows:

In the topological world we can still take a get a tubular neighbourhood of ΔE inside of $E \times E$, but now because it's not smooth this is just a topological $\mathbb{R}^{\dim E}$ bundle over E, which we denote TE. We can also build a vertical version $T^v E$.

1.2. **Tangential Structure.** So as before we add a notion of tangential structure. Given a map $\theta : X \to BO(d) \times B$ a θ-tangential structure on $E \to B$ is a map ℓ making the diagram

$$\begin{tikzcd} & X \arrow[d, "{(\xi,p)}"] \\ E \arrow[ur, dashed, "\ell"] \arrow[r, "{(\varepsilon \oplus T^v E, \pi)}"'] & BO(d) \times B \end{tikzcd}$$

commute. This is equivalent to a diagram

$$\begin{tikzcd}
T^v E \arrow[r, dashed] \arrow[d] & V \arrow[d, "\xi"] \\
E \arrow[r, dashed] \arrow[dr, "\pi"'] & X \arrow[d, "p"] \\
& B
\end{tikzcd}.$$

Where the top map is a map of vector bundle map. This we can generalize of $E \xrightarrow{\pi} B$ being topological manifolds, but now the top map is a map of microbundles, i.e. a fibrewise homeomorphism.

1.3. **Cobordism Categories.**

Now we can define $\mathrm{Cob}^\delta(\theta)$ have objects to be

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "\pi"] & B \times \{a\} \times \mathbb{R}_+ \times \mathbb{R}^\infty \arrow[d] \\
B \arrow[r, hook] & B \times \{a\}
\end{tikzcd}$$

with tangential structure as described. The morphisms are given by Cobordisms with tangential structure which pullback appropriately along inclusion.

2. Bivariant Theories and the Categorified Index Theorem Theorem

2.1. **Bivariant Theories.**

We define a category with BIV objects given by triples

$$V \xrightarrow{\xi} X \xrightarrow{p} B$$

as in the previous section. The morphisms in this category are harder to describe. Given a map $g : B' \to B$ we can pullback the bundles. Then a map $(p', \xi', B') \to (p, \xi, B)$ is given by a map $(p', \xi') \to g^*(p, \xi)$ over B'.

Definition 2.1. A *bivariant theory* with values in a category $\mathcal{C}$ is a functor $\mathrm{BIV} \to \mathcal{C}$ such that this functor is homotopy invariant in both X and E.

This is bivariant in the sense that for a morphism on the base we get a contravariant map in the target category, and for a morphism on the fibre we get a covariant map.

Example 2.2. The bivariant A-theory functor didn't use anything about smoothness so it is still a bivariant theory. This is a spectra valued theory

Example 2.3. $\mathrm{Cob}^\delta(-)$ is a bivariant theory with values in categories. As in the previous talk, this can be upgraded from a bivariant theory $B\mathrm{Cob}(-)$ with values in spaces.

2.2. **Assembly and Coassembly.**

As described previously, for any bivariant theory C we have a map $\nabla_C : C \to \overline{C}^{\&}$ where $\overline{C}^{\&}$ is the best contravariantly exissive approximation to C, i.e. restrict C over a point to get $\overline{C}$ and then extend to get $\overline{C}^{\&}$. Further, this $\overline{C}$ has a excissive approximation $\overline{C}^{\%}$ and there is an assembly map $\alpha_C : \overline{C}^{\%} \to \overline{C}$.

Remark 2.4. If F is contravariantly excessive with image in spaces or spectra, we can identify F with the space or spectrum $\Gamma(F_B(X) \to B)$. Examples of these kinds of section include the A-theoretic Euler characteristic. This is why it makes sense to study A-theory as a bivariant theory in our context.

2.3. **The Categorified Index Theorem.**

Theorem 2.5. *For bivariant theories C and D with values in spaces or spectra with and $\overline{C}$ is excissive, then for any map of bivariant theories $\tau : C \to D$ we have a commutative diagram*

$$\begin{tikzcd} C \arrow[rr, "\tau"] \arrow[d, "\nabla_C"'] & & D \arrow[d, "\nabla_D"] \\ \overline{C}^{\&} \arrow[r, "\tau^{\%}"'] & (\bar{D}^{\%})^{\&} \arrow[r, "\alpha^{\&}_{\bar{D}}"'] & \overline{D}^{\&} \end{tikzcd}.$$

Proof. Trivial. □

3. Recovering The Index Theorem

3.1. **The Generalized Index Theorem.** We will now construct a map of bivariant theories

$$\tau : \Omega B\mathrm{Cob} \to A$$

to which we will apply the categorified index theorem. We will not go through the details of defining this map but loosely it sends an n-tuple of composable d-dimensional coboridsms over B together with their tangent bundle to their union over B.

To prove the generalized index theorem we will need the following as input.

Theorem 3.1 (Gomez-Lopez Kupers)**.** *The functor $\Omega\overline{B\mathrm{Cob}}$ is excissive.*

What Gomez-Lopez Kupers Prove. Loosely, it is proved that $\overline{B\mathrm{Cob}}(\xi)$ is naturally equivalent to a spectrum $B(\xi)$, a space which is a configuration space of manifolds with ξ tangential structure embedded in high dimensional euclidian space. This is done by applying smoothing theory to GMTW. □

Remark 3.2. Our understanding of this $\Omega\overline{B\mathrm{Cob}}$ is not as good as in the smooth case, for example we don't understand the entire homotopy type.

Theorem 3.3. *There is a commutative diagram*

$$\begin{tikzcd} B\mathrm{Cob} \arrow[rr, "\tau"] \arrow[d, "\nabla_{B\mathrm{Cob}}"'] & & A \arrow[d, "\nabla_D"] \\ \overline{B}\mathrm{Cob}^{\&} \arrow[r, "\tau^{\%}"'] & (\bar{A}^{\%})^{\&} \arrow[r, "\alpha^{\&}_{\bar{D}}"'] & \overline{A}^{\&} \end{tikzcd}.$$

Proof. Apply the categorified theorem. □

3.2. **The Classical Index Theorem.** We will now go through the somewhat arduous process of recovering the classical theorem.

Corollary 3.4. *For a bundle of topological manifolds $E \xrightarrow{\pi} B$, there is a commutative diagram*

$$\begin{tikzcd} & A^{\%}_B \arrow[d, "\alpha_{\bar{A}}"] \\ B \arrow[ur, "\chi^{\%}(\pi)"] \arrow[r, "\chi(\pi)"'] & A_B(E) \end{tikzcd}.$$

Proof Sketch. Define $\theta = (T^vE \to E \xrightarrow{\pi} B)$. We can identify $\chi(\pi)$ and $\chi^{\%}(\pi)$ as elements in $\pi_0(\overline{A}^{\&}(\theta))$ and $\pi_0((\overline{A}^{\%})^{\&}(\theta))$ respectively.

Post composing with α on sections is exactly the map $(\bar{A}^{\%})^{\&} \xrightarrow{\alpha^{\&}_{\bar{D}}} \overline{A}^{\&}$ in the above diagram. So to prove the theorem all we need to do is product an element of $\pi_0(\Omega B\mathrm{Cob}(\theta))$ and verify that the diagram maps it to the correct place.

We get a canonical manifold bundle given by θ itself and by taking the product of this bundle with the interval we get a cobordism. We denote by $[\pi]$ the associated element in $\pi_0(\Omega B\mathrm{Cob}(\theta))$. We just need to check that $\nabla_A \circ \tau$ takes this to $\chi(\pi)$ (then we also get it for a point and thus we will as get that $\tau^{\%} \circ \nabla_{B\mathrm{Cob}}$ takes this to $\chi^{\%}(\pi)$). Now $\tau([\pi])$ is just $E \sqcup E \to E$. Looking under coassembly this gives exactly $\chi(\pi)$. $\square$