Note: These notes mostly follow A Primer on Mapping Class Groups, by Benson Farb and Dan Margalit, adapted for a talk.

Mapping Class Groups

The collection of homeomorphisms Homeo(X) of a topological space X is a group under composition. We can define and equivalence relation $\sim$ on this group, by saying that two homeomorphisms are equivalent if there is a homotopy between them which is a homeomorphism at all times (this type of homotopy is called an isotopy).

This equivalence relation of isotopy respects the group structure because we can compose isotopies, so the quotient $\mathrm{Homeo}(X)/\sim$ is a group which we will call the mapping class group MCG(X).

Today we will deal with oriented 2-manifolds and we will restrict ourselves to orientation preserving, boundary fixing homeomorphisms.

Technicalities

Generally one has to be careful about distinguishing homotopy, isotopy, and smooth isotopy (which we will not define), but in dimensions 1 and 2 these all coincide (so long as things are oriented).

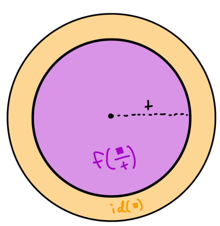
A completely non-obvious technical result is called the isotopy extension principal. If we have an isotopy of some simple proper arc or simple closed curve inside our surface, then we can actually extend this to the entire surface.

Lastly, if our topological space is punctured we often think of these punctures as simply marked points that must be preserved.

Disks

Suppose that f is a homeomorphism. Then at time t we define

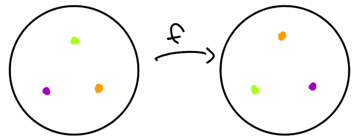
$$f_t(x) = \begin{cases} f(tx)/t & x \leq t \\ \mathrm{id} & \text{otherwise} \end{cases}.$$

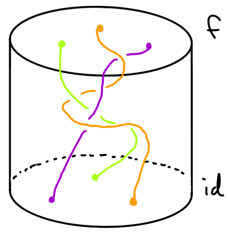


This is an isotopy of the identity map to f. This shows that MCG(D^2) is contractible and is called Alexander's trick.

Punctured 2-Disks

Here we consider the closed unit disk D^2 with n punctures in its interior. Let f be a a homeomorphism of $D^2 - \{p_1, \ldots, p_n\}$. f then extends to a homeomorphism of all of D^2 that interchanges the marked points. Then there is an isotopy F of f to the identity in D^2. Then the preimage of $(p_1, \ldots, p_n)$ under F gives a braid embedded in $D^2 \times I$.





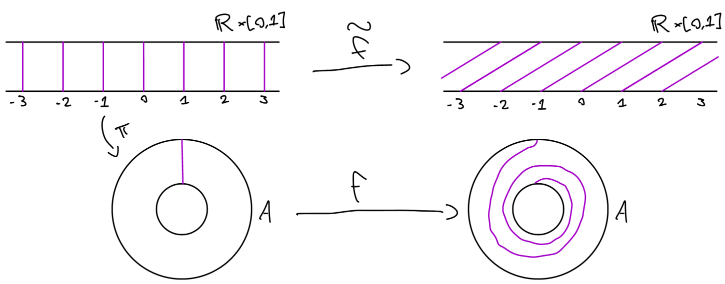
If the braid is not trivial then we could not isotope the map to the identity without moving the points so we get an injective map $\mathrm{MCG}(D^2 - \{p_1, \ldots, p_n\}) \to B_n$ (it is clear that composition is the same). On the other hand, given a braid we can embed it in $D^2 \times I$ and find a map D^2 to D^2 so we see that $\mathrm{MCG}(D^2 - \{p_1, \ldots, p_n\}) \cong B_n$.

We will see a more explicit way to show surjectivity later when we discuss Dehn twists.

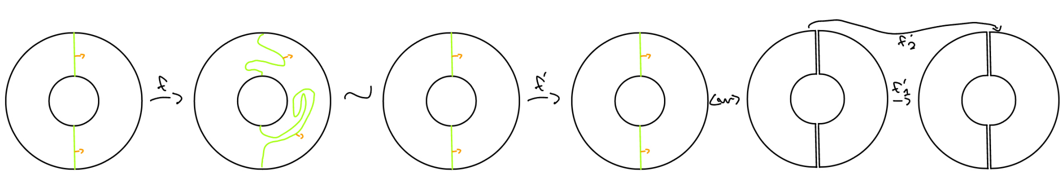
The Annulus

Let A be the annulus, whose universal cover is $\tilde{A} = \mathbb{R} \times [0, 1]$. For any $f : A \xrightarrow{\sim} A$ there is a preferred lift of f to $\tilde{f} : \tilde{A} \to \tilde{A}$ which fixes $(0, 0)$. Now since this is a lift of some a boundary preserving map on A, restricting to $\mathbb{R} \times \{1\}$ we get a map $x \mapsto x + n$ for $n \in \mathbb{Z}$. This allows us to define a map $\Phi : \mathrm{MCG}(A) \to \mathbb{Z}$ and its clear that this is a group homomorphism.

It is very easy to see that this map is surjective.



Now we want to see that this is injective. Suppose that f maps to id under Φ. Then we can just isotope arcs cutting our annulus into two disks to themselves. Since f was orientation preserving the two disks are sent to themselves and we use our result for D^2.



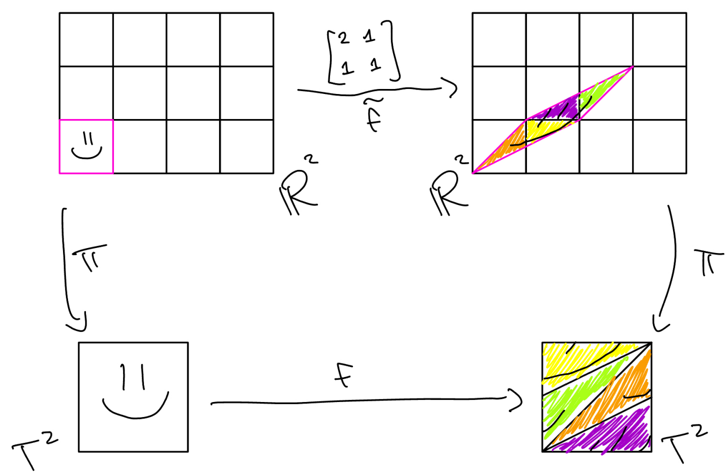
The element corresponding to -1 is called a Dehn twist. Whenever we have a curve embedded in a surface we can twist around its neighbourhood annulus.

The Torus and the Punctured Torus

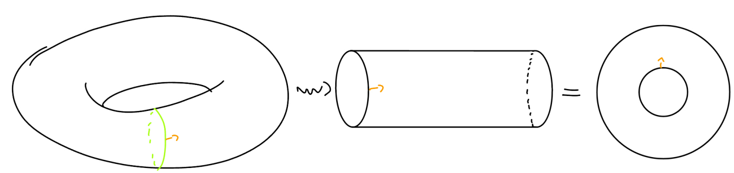
For this we note a concept of general utility. Given a homotopy invariant such as homology or homotopy groups, there is an action of the mapping class groups because isotopy is a stronger equivalence relation than homotopy.

Let $f : T^2 \to T^2$. Then there is an induced action on homology $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ so we have a map $\Phi : \mathrm{MCG}(T^2) \to \mathrm{GL}(2,\mathbb{Z})$. The generators for H^2 are the simple closed curves α and β corresponding to $(0,1)$ and $(1,0)$, which have intersection number 1. The intersection number (which must be preserved) is the determinant so we get that Φ is actually a map to $\mathrm{SL}(2,\mathbb{Z})$.

To see that Φ is surjective we can just take $\tilde{f} \in \mathrm{SL}(2,\mathbb{Z})$ which is a homeomorphism of $\mathbb{R}^2$, the universal cover of T^2. The descends because a point doesn't depend on its preimage and it has an inverse.



Suppose that f maps to $\mathrm{id} \in \mathrm{SL}(2,\mathbb{Z})$, then we know that if α and β are two closed curves we know that $f_*\alpha$ is isotopic to α and we can just extend this isotopy to all of T^2 so α is fixed. Then since the orientation is fixed so we can cut along α to achieve an annulus and apply the previous result for the annulus. If we keep track of fixed points this also works for the once punctured torus.



Alexander's Method

This method that we've seen for T^2, $S_{1,1}$, and A generalize to a universal strategy for understanding mapping class groups of surfaces. Cut up the surface into disks and once marked disks using some sufficiently nice simple proper arcs and essential simple curves, and then use the result for the disk. One has to be a little bit careful, but this does work in general.

Action on Teichmüller Space

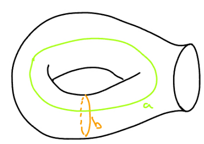
The Teichmüller space $T(S)$ is composed of pairs (ϕ, X) where $\phi : S \to X$. We have an action of $\mathrm{MCG}(X)$ on $T(S)$ by pre-composition.

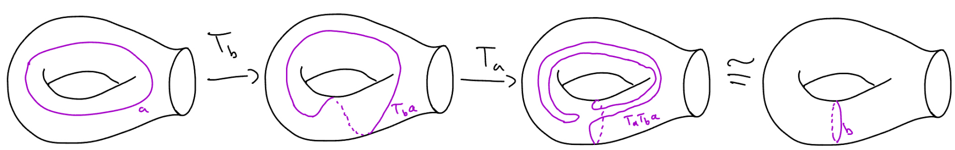
A Technical Lemma and the Braid Relation

Given the homotopy class of an essential simple closed curve a denote the Dehn twist around a as T_a. T_a as a map depends on the representative for a, but as an element of the mapping class group it does not. We remark that $T_a = T_b \iff a = b$. It is clear that $a = b \implies T_a = T_b$. To verify the reverse implication we can just find some essential curve c that intersects a and not b and verifying that $\iota(T_a c, c) \neq 0$ whereas twisting around $\iota(T_b c, c) = 0$.

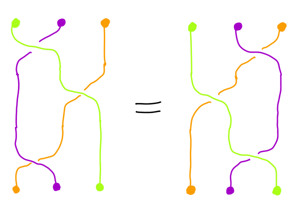
We note that for a homeomorphism f, $T_{f(a)} = f T_a f^{-1}$.

Now we will prove the braid relation. For two essential simple closed curves a and b we have $T_a T_b T_a = T_b T_a T_b$. Perhaps this relation is better written as $(T_a T_b) T_a (T_a T_b)^{-1} = T_b$, or even better yet that $T_{T_a T_b(a)} = T_b$. So we just need to check that $T_a T_b(a) = b$. This can be seen from a picture.





The name for this relation comes from the same relation in B_n.



The mapping class group can of a surface can be seen to satisfy many other relations relations and they all have nice associated pictures.

Generating the Mapping Class Group

As a final remark we'll note that mapping class groups are finitely generated. For a closed surface these generators are sufficient (there are even smaller sets). This particular set is due to Lickorish.