K-THEORY OF FINITE FIELDS

SACHA GOLDMAN

The goal of these notes is to compute $K_*(\mathbb{F}_q)$, originally computed originally by Quillen [Qui72]. We will follow (at least for the beginning) the notes Shay Ben-Moshe [BM20], which uses the modern $+$-Construction given by Nikolaus [Nik17]. Because these are talk notes, the proofs vary between fully detailed and very loose sketches.

Contents

1. Introduction

Thus far in the seminar we have worked on computing $H^*(\mathrm{GL}_\infty\mathbb{Z};\mathbb{Q})$ using mostly analytic techniques. We will now switch gears to working on calculating $H^*(\mathrm{GL}_\infty\mathbb{Z};\mathbb{Z}/p)$, which will combine with our earlier results to compute $H^*(\mathrm{GL}_\infty\mathbb{Z})$.

In this talk we will discuss the K-theory of rings, with the perspective that K-groups of a ring R are the homotopy groups of some space (or really spectrum) $K(R)$, with the notable property that for the rings we are interested in

$$H^*(K(R)) = H^*(\mathrm{GL}_\infty R).$$

We will let $\mathbb{F}_q$ denote the finite field with $q = p^r$ elements and we will compute the homotopy groups of $K(\mathbb{F}_q)$. This will serve as a building block as we move towards a computation of the cohomology of some p-localization of $\mathrm{GL}_\infty(\mathbb{Z})$.

2. Connective K-Theory and Quillen's $+$-Construction

We will begin with an extremely terse introduction to K-theory. Let $\mathcal{S}$ be the ∞-category of spaces (all categories will be ∞-categories). Then there is an an adjunction pair

$$\mathrm{fgt} : \mathrm{Ab}(\mathcal{S}) \rightleftharpoons \mathrm{CMon}(\mathcal{S}) : \mathrm{gpc}$$

analogous to the standard adjunction between abelian groups and commutative monoids. In other words, this group completion is defined as to satisfy the usual universal property of the group completion. We will give a more computable definition of this functor.

Definition 2.1. Given a ring R we define

$$K(R) = (\mathrm{Proj}_R^{\simeq}, \oplus)^{\mathrm{gpc}}$$

where Proj_R is the category of finitely generated projective modules over R.

If we view the above definition 1-categorically (i.e. with the standard group completion), then the above is exactly the definition of $K_0(R)$. From a modern perspective this is the "obvious" generalization of K_0. The assumption in the next definition hints at the defect of this definition of $K(R)$.

Definition 2.2. For $n \geq 0$ define

$$K_n(R) = \pi_n(K(R)).$$

Because $K(R)$ is an object in $\mathrm{Ab}(\mathcal{S})$, the homotopy groups of $K(R)$ are all abelian.

Aside 2.3. The category $\mathrm{Ab}(\mathcal{S})$ is isomorphic to $\mathcal{S}\mathrm{p}_{\geq 0}$, the category of connected spectra. We should for this reason really be writing $K^{\mathrm{conn}}(R)$ because this is the connective truncation of the spectrum $K(R)$ (which has negative homotopy groups), but we will ignore these technicalities.

To compute K-theory groups we require a more explicit construction of the functor gpc.

Definition 2.4. For M an object of $\mathrm{CMon}(\mathcal{S})$ take $x \in \pi_0(M)$. Then define

$$M_x = \mathrm{colim}(M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots).$$

Then define M_I inductively as $M_I = (M_{I \setminus \{x\}})_x$ where $x \in I$ and $M_\emptyset = M$. Then finally define

$$M_\infty = \operatorname*{colim}_{\substack{I \subseteq \pi_0(M) \\ I \text{ finite}}} M_I.$$

If M were simple a commutative monoid in the category of sets, then in the above construction M_x would be $M[1/x]$ and $M_\infty = M^{\mathrm{gpc}}$. Indeed, in our case it is clear that $\pi_0(M_\infty) = \pi_0(M)^{\mathrm{gpc}}$ because π_0 commutes with colimits.

Unfortunately, $M_\infty \neq M^{\mathrm{gpc}}$. For example, we are interested in computing $(\mathrm{Proj}_R^{\simeq}, \oplus)_\infty$. When we are working over a field $\mathbb{F}$ the monoid $\mathrm{Proj}_R^{\simeq}$ is simply vector spaces, and so $\mathrm{Proj}_{\mathbb{F}}^{\simeq} = \coprod B\mathrm{GL}_n(\mathbb{F})$ (where $\mathrm{GL}_n(\mathbb{F})$ is thought of as a discrete group). The colimit then is just $\mathbb{Z} \times B\mathrm{GL}_\infty \mathbb{F}$. This does not have abelian fundamental group (usually) and so cannot be $(\mathrm{Proj}_{\mathbb{F}}^{\simeq}, \oplus)^{\mathrm{gpc}}$.

Proposition 2.5. *The map $M_\infty \to M^{\mathrm{gpc}}$ is an equivalence exactly when $\pi_1(M)$ is hypoabelian, i.e.the maximal perfect subgroup of $\pi_1(M)$ is 1 (this also actually means that $\pi_1(M)$ is abelian).*

Theorem/Definition 2.6. The inclusion functor of the subcateogry of spaces with hypoableian fundamental group $\mathcal{S}^{\mathrm{hypo}} \to \mathcal{S}$ has a left adjoint $(-)^+$.

Proof. We just check that the forgetful functor commutes with limits and filtered colimits and then it is guaranteed to have a left adjoint by the adjoint functor theorem.

For limits it is enough to check closure under products and pullbacks. This can easily be checked and, in fact, their nice extension properties are the reason we work with hypoabelian groups.

For filtered colimits one can use that the homotopy groups of a filtered colimit are the colimit of the homotopy groups. So it's enough to check that the category of hypoabelian groups has filtered colimits. The inclusion of hypoabelian groups into groups has a left adjoint given by hypoabelianization (quotienting by the terminal term in the transfinite derived series), and so hypoabelian groups must be closed under filtered colimits. ∎

Proposition 2.7. *The map $X \to X^+$ is a homology equivalence. This is our version of the classical McDuff-Segal theorem.*

Proof. Using the adjunction we get that $\mathrm{Map}(X^+, B^n\mathbb{Z}) = \mathrm{Map}(X, B^n\mathbb{Z})$. Applying π_0 gives us a cohomology equivalence and then we just use the universal coefficient theorem. $\square$

Theorem 2.8.

$$(M_\infty)^+ \simeq M^{\mathrm{gpc}}.$$

Proof. We have the square

$$\begin{tikzcd} (M_\infty)^+ \arrow[r] \arrow[d] & M^{\mathrm{gpc}} \arrow[d] \\ (M^+)_\infty \arrow[r] & (M^+)^{\mathrm{gpc}} \end{tikzcd}.$$

First one verifies that the bottom map makes sense by showing that $(-)^+$ commutes with products using the Yoneda lemma.

The bottom map is then an equivalence because M^+ has abelian π_1. The right map is an equivalence because things were defined to be adjoints. To show that the left map is an equivalence with use the Yoneda lemma and play a game with commuting colimts and $\mathrm{Map}(-, X)$. The result follows. $\square$

3. Adams Operations and $B\mathrm{U}^{\psi^q}$

To understand the algebraic K-theory of finite fields, we aim to compare it to something we already understand, the topological K-theory of $\mathbb{C}$. We will do this using the remarkable map

$$B\mathrm{GL}_\infty(\mathbb{F}_q) \to BU^{\psi^q}.$$

But, before we construct this map we should understand the codomain.

To begin, we recall that the topological K-theory of a space X is given by taking the category $(\mathrm{Vect}_{\mathbb{C}}^{\simeq,\mathrm{top}}(X), \oplus)$ of vector bundles over X (here we do care about topology) and taking the group completion, i.e.

$$KU(X) = (\mathrm{Vect}_{\mathbb{C}}^{\simeq,\mathrm{top}}, \oplus)^{\mathrm{gpc}}.$$

This is a generalized cohomology theory, and so we can that $KU(X) \cong \mathrm{Map}(X, KU(*))$. We are interested in

$$(\mathrm{Vect}_{\mathbb{C}}^{\simeq,\mathrm{top}}(*), \oplus)^{\mathrm{gpc}}$$

which only differs from our earlier computation in that we now care about topology. So we get that this is $(\mathbb{Z} \times B\mathrm{U}_\infty)^+$, but $\pi_1(B\mathrm{U}_\infty) = \pi_0(\mathrm{U}_\infty) = 0$ so we know that $\mathbb{Z} \times B\mathrm{U}$ is actually $KU(*)$.

We define KU^n analogously to K_n and similarly $KU^0(X)$ coincides with the usual definition of the 1-categorical group completion of the monoid of vector bundles. Notably, KU^0 is actually a ring using the tensor product of vector bundles

Theorem/Definition 3.1 (Adams operations)**.** Suppose that R is either the semi-ring $(\mathrm{Vect}_{\mathbb{C}}^{\simeq}(X), \oplus)$ (viewed 1-categorically), a representation semi-ring $\mathrm{Rep}_{\mathbb{F}}(G)$, or the group completion of one of these semi-rings. These rings are part of a more general class of rings called λ-semi-rings, notably because they have the extra structure of exterior powers. Then, R has a series of semi-ring endomorphisms ψ^k for $k \in \mathbb{N}$ called *Adams operations* with the following properties (which determine them uniquely)

(1) For "1-dimensional objects" (line bundles, 1-dimensional representations) $\psi^k(E) = E^k$.
(2) These operations are functorial (in X or G repectively).

These operations provide a nice way of packaging the structure of exterior powers on the topological K-theory rings and on representation rings. Maps which preserve the exterior power structure preserve the ψ^k's.

We remark that as cohomology operations these are realized by maps $BU \to BU$ by the Yoneda lemma. In particular BU^{ψ^q} is defined as the fibre of the map $\psi_q - \mathrm{id} : BU \to BU$. To fully understand this fibre, we have the following.

Theorem 3.2.

$$\pi_n(B\mathrm{U}^{\psi^q}) = \begin{cases} 0 & n = 2k \\ \mathbb{Z}/(q^k - 1) & n = 2k - 1 \end{cases}.$$

Proof.

Theorem 3.3 (Bott Periodicity)**.** *$\pi_{2k-1}(BU) = 0$ and $\pi_{2k}(BU) = \widetilde{KU}^0(S^{2k}) \cong \mathbb{Z}$ (where the last isomorphism is only of abelian groups). In the case of $k = 1$ this group is generated by $L - 1$ where L is the canonical line bundle over $S^2 \cong \mathbb{C}P^1$ with $(L-1)^2 = 0$. For $k \geq 0$ the generator for $\widetilde{KU}^0(S^{2k}) = \widetilde{KU}^0((S^2)^{\wedge k}) = \widetilde{KU}^0(S^2)^{\otimes k}$ is $(L-1)^{\otimes k}$.*

Now we note that

$$\psi^q((L-1)^{\otimes k}) = (L^q - 1)^{\otimes k} = ((L^q - 1 + 1) - 1)^{\otimes k} = (q(L-1))^{\otimes k} = q^k (L-1)^{\otimes k}.$$

Then the long exact sequence for the fibration defining $(BU)^{\psi^q}$ simplifies to

$$0 \to \pi_{2n}(B\mathrm{U}^{\psi^q}) \to \mathbb{Z} \xrightarrow{q^n - 1} \mathbb{Z} \to \pi_{2n-1}(B\mathrm{U}^{\psi^q}) \to 0.$$

so we can compute the desired result.

□

4. K-Theory of Finite Fields

We can now finally turn our attention to $\mathbb{F}_q$ where $q = p^r$. To begin, we have the composition

$$\mathrm{GL}_1(\bar{\mathbb{F}}_p) = \operatorname{colim} \mathbb{F}_{p^k}^\times \cong \operatorname{colim} \mathbb{Z}/(p^k - 1) = (1/\ell\mathbb{Z})/\mathbb{Z}$$

where ℓ is the collection of all elements coprime to p. Then $(1/\ell\mathbb{Z})/\mathbb{Z}$ lives inside $\mathrm{U}(1)$ as roots of unity. We will call this composition $\sigma : \mathbb{Z}_\ell \to \mathrm{U}(1)$. Our goal is to upgrade σ the map mentioned at the begin of the previous section.

Let $\mathrm{Rep}_{\mathbb{F}}(G)$ denote the representation semi-ring of G over $\mathbb{F}$ and let $R_{\mathbb{F}}(G)$ be its group completion. Now we will use the following.

Theorem 4.1 ([Gre55])**.** *Given an element of $\rho \in \mathrm{Rep}_{\bar{\mathbb{F}}_p}(G)$ define*

$$\begin{aligned} \chi_\rho : G &\to U(1) \\ g &\mapsto \sum_{\lambda \in \mathrm{eig}(\rho(g))} \sigma(\lambda) \end{aligned}$$

Then χ_ρ is the character of virtual complex representation, i.e. we obtain a map $\mathrm{Rep}_{\bar{\mathbb{F}}_p}(G) \to R_{\mathbb{C}}(G)$.

Now we have a natural map

$$\mathrm{Map}(BG, \bigsqcup B\mathrm{U}(n))^{\mathrm{gpc}} \to \mathrm{Map}(BG, (\bigsqcup B\mathrm{U}(n))^{\mathrm{gpc}})$$

And taking π_0 gives a map

$$R_{\mathbb{C}}(G) \to KU^0(BG).$$

So we have a composition

$$\mathrm{Rep}_{\mathbb{F}_q}(G) \to \mathrm{Rep}_{\bar{\mathbb{F}}_p} \to R_{\mathbb{C}}(G) \to KU^0(BG)$$

and it is easily seen that each map preserves dimension and exterior powers, and so it also preserves the Adams operations.

Proposition 4.2. *For the characters of virtual representations $\psi^k(\chi)(g) = \chi(g^k)$.*

Proof. Use additivity to restrict to representations. Then look at the subgroup generated by g. Then decompose as one dimensional representations. □

Lemma 4.3. *If $\rho \in \mathrm{Rep}_{\mathbb{F}_q}(G)$ then $\psi^q(\chi_\rho) = \chi_\rho$.*

Proof. We have the Frobenius map Frob^q which fixes ρ. Then we compute

$$\chi_\rho(g) = \chi_{\mathrm{Frob}^q\rho}(g) = \sum_{\lambda \in \mathrm{eig}(\mathrm{Frob}^q\rho(g))} \sigma(\lambda) = \sum_{\lambda \in \mathrm{eig}(\rho(g^q))} \sigma(\lambda) = \chi_\rho(g^q) = \psi^q\chi_p(g)$$

□

Now $KU^0(BG) = \pi_0(\mathrm{Map}(BG, \mathbb{Z} \times B\mathrm{U}))^{\psi^q} = \pi_0(\mathrm{Map}(BG, \mathbb{Z} \times (B\mathrm{U})^{\psi^q}))$ using the fact that $KU^1(BG) = 0$ (as given by the Atiyah-Segal theorem). So we have a map $\mathrm{Rep}_{\mathbb{F}_q(G)} \to \pi_0(\mathrm{Map}(BG, B\mathrm{U}^{\psi^q}))$ and taking $G = \mathrm{GL}_n(\mathbb{F}_q)$ and taking the standard representation of $\mathrm{GL}_n(\mathbb{F}_q)$ we get maps $\mathrm{GL}_n(\mathbb{F}_q) \to B\mathrm{U}^{\psi^q}$ which are compatible so we get a total map

$$\theta : \mathrm{GL}_\infty(\mathbb{F}_q) \to B\mathrm{U}^{\psi^q}.$$

Now using the universal property of the + construction we get the induced map

$$\begin{tikzcd} B\mathrm{GL}_\infty(\mathbb{F}_q) \arrow[r, "\theta"] \arrow[d] & B\mathrm{U}^{\psi^q} \\ B\mathrm{GL}_\infty(\mathbb{F}_q)^+ \arrow[ur, dashed] & \end{tikzcd}.$$

Now the bottom map is a map of simple spaces (spaces with π_1 abelian acting trivially on higher homotopy groups). This is because all spaces in $\mathrm{Ab}(\mathcal{S})$ are simple and this follows from the construction for $B\mathrm{U}^{\psi^k}$. We would like to apply the following theorem.

Theorem 4.4 (Whitehead++)**.** *A homology equivalence $X \to Y$ of simple spaces is a homotopy equivalence.*

In spectacular form, Quillen uses this to prove the following.

Theorem 4.5 ([Qui72])**.**

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 2k \neq 0 \\ \mathbb{Z}/(q^k - 1) & n = 2k - 1 \end{cases}.$$

Proof. We just need to prove that θ is a homology isomorphism. Then we will get that $K_n(\mathbb{F}_q) = \pi_n(\mathbb{Z} \times B\mathrm{U}^{\psi^q})$.

We will show that this is a homology isomorphism rationally and then one prime at a time. To begin we note that $B\mathrm{U}^{\psi}$ is both rationally contractible and p-locally contractible, so we should show that these cohomology groups vanish for $B\mathrm{GL}_\infty(\mathbb{F}_q)$ too.

$B\mathrm{GL}_\infty(\mathbb{F}_q)$ is rationally contractible because

$$H^*(B\mathrm{GL}_\infty(\mathbb{F}_q);\mathbb{Q}) = H^*(\mathrm{colim}\, B\mathrm{GL}_n(\mathbb{F}_q);\mathbb{Q}) = \lim H^*(B\mathrm{GL}_n(\mathbb{F}_q);\mathbb{Q})$$

but everything in the limit in zero because all the groups are finite, so this is just 0.

Now we will show that $H^*(B\mathrm{GL}_\infty(\mathbb{F}_q);\mathbb{Z}/p)$ vanishes. To begin we will show that $H^*(B\mathrm{GL}_n(\mathbb{F}_q);\mathbb{Z}/p)$ vanishes for $* < r(p-1)$. Let $\mathrm{UT}_n(\mathbb{F}_q)$ denote the subgroup of upper triangular matrices. This includes into $\mathrm{GL}_n(\mathbb{F}_q)$ as a p-sylow subgroup and so it induces an injection on cohomology mod-p. So it's enough to show that $H_*(\mathrm{UT}_n(\mathbb{F}_q);\mathbb{Z}/p)$ vanishes in the desired range. Now we use induction on n. The base case is immediate. For the inductive step we have the extension

$$\mathbb{F}_q^n \ltimes \mathbb{F}_q^\times \to \mathrm{UT}_{n+1}(\mathbb{F}_q) \to \mathrm{UT}_n(\mathbb{F}_q)$$

where $\mathbb{F}_q^n \rtimes \mathbb{F}_q^\times$ is the subgroup of matrices with only the top row and 1's along the remainder of the diagonal. Then one shows the vanishing result for the groups above and applies the Hochschild-Serre spectral sequence.

So we know that $H^*(B\mathrm{GL}_\infty(\mathbb{F}_q);\mathbb{Z}/p)$ vanishes for $* < r(p-1)$ and we would like to upgrade our result. For this we will use transfers in K-theory. We only discuss the transfer we need but this outlines a more general strategy. We know that we have a field extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^s}$ giving a map

$$K\mathbb{F}_q \to K\mathbb{F}_{q^s}.$$

There is also a map

$$(\mathrm{Proj}^{\cong}_{\mathbb{F}_{q^s}}, \oplus) \to (\mathrm{Proj}^{\cong}_{\mathbb{F}_q}, \oplus)$$

given by just forgetting the fact that the vector space was over the bigger field. This gives a map

$$K\mathbb{F}_{q^s} \to K\mathbb{F}_q.$$

It is easy to check that the composition

$$K\mathbb{F}_q \to K\mathbb{F}_{q^s} \to K\mathbb{F}_q.$$

is just multiplication by s. Take s coprime to p. Then localizing at p the composition is the identity so it is the identity on cohomology, but the cohomology of the middle term vanishes in a larger range. Taking s large enough we can show that any cohomology vanishes.

Now we need to tackle cohomology at ℓ a prime not p. Given that this is where the cohomology is concentrated, it shouldn't be a surprise that this is by far the hardest part. This computation is very hands on, and we only give an extremely rough outline.

Step 1

Use the diagram

$$\begin{tikzcd} B\mathrm{U}^{\psi^q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & BU^{[0,1]} \arrow[d] \\ B\mathrm{U} \arrow[r, "(\mathrm{id},\psi^q)"'] & B\mathrm{U} \times B\mathrm{U} \end{tikzcd}$$

to compute (using the Eilenberg-Moore Spectral Sequence) that

$$H^*(B\mathrm{U}^{\psi^q};\mathbb{Z}/\ell) \cong P[c_{kr}] \otimes \Lambda[e_{kr}]$$

as abelian groups. Here k runs over all natural numbers, and $\deg(c_{kr}) = 2kr$ and $\deg(e_{kr}) = 2kr-1$.

Step 2

Find explicit characterizations for these c_{kr}'s and e_{kr}'s. For the c_{kr}'s these will come from pulling back churn classes, and for the e_{kr}'s the process is more complicated.

Step 3

The group $C \cong \mathbb{F}_q(\sqrt[\ell]{1})^\times$ has a natural action on $\mathbb{F}_q(\sqrt[\ell]{1})$. So we have a map

$$C \hookrightarrow \mathrm{GL}_s(\mathbb{F}_q).$$

Then using our earlier techniques we get the composition on cohomology

$$H^*(B\mathrm{U}^{\psi^q};\mathbb{Z}/\ell) \to H^*(\mathrm{GL}_s(\mathbb{F}_q);\mathbb{Z}/\ell) \to H^*(C;\mathbb{Z}/\ell)^{\mathrm{Gal}(\mathbb{F}_q(\sqrt[\ell]{1}),\mathbb{F}_q)}.$$

Then one inspects what this map does to the characterizations of the generators from earlier. Now take n to be some integer and take $n = sm + c$ with c minimal. Then we have a map

$$S_m \ltimes (\mathrm{Gal}(\mathbb{F}_q(\sqrt[\ell]{1}),\mathbb{F}_q) \ltimes C)^m) \to \mathrm{GL}_n(\mathbb{F}_q)$$

where we add trivial representations of $\mathbb{F}_q$ if needed. Then it turns out that the map

$$H^*(\mathrm{GL}_n(\mathbb{F}_q);\mathbb{Z}/\ell) \to H^*(C^m)^{\Sigma_m \ltimes \mathrm{Gal}(\mathbb{F}_q(\sqrt[\ell]{1}),\mathbb{F}_q)^m}$$

will be a cohomology isomorphism. Using our understanding of where the classes from $B\mathrm{U}^{\psi^q}$ we get that a cohomology isomorphism in increasingly higher degrees as we take $n \to \infty$.

$\square$

References

[BM20] S. Ben-Moshe, *Algebraic k-theory of finite fields*, 2020, available online at https://shaybm.com/k-theory-seminar/notes/k-theory-of-fin-fields.pdf. 1

[Gre55] J. A. Green, *The characters of the finite general linear groups*, Transactions of the American Mathematical Society, **80** (1955) (2), 402–447. 4

[Nik17] T. Nikolaus, *The group completion theorem via localizations of ring spectra*, 2017, available online at https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Group_completion.pdf. 1

[Qui72] D. Quillen, *On the cohomology and k-theory of the general linear groups over a finite field*, Annals of Mathematics, **96** (1972) (3), 552–586. 1, 5