Defining Higher Homotopy Groups + Introduction

As a set $\pi_1(X,x)$ is based homotopy classes of maps $S^1 \to X$ such that 0 maps to x. We define $\pi_n(X,x)$ in the same way, as homotopy classes of maps $S^n \to X$ sending some special point N to x.

To make this into a group, we do the same thing as for π_1, that is we consider (S^n, N) as $(I^n/\partial I^n, \partial I^n/\partial I^n)$ then we can glue two cubes together along a pair of opposite faces, where the maps agree, to get another copy of S^n and a map from it to X.

Unlike π_1 these groups are abelian. To see this we can take our two connected copies of I^n and put them into a bigger n-cube. We can map this to X using the maps we already have on the inner copies of I^n and map the remainder to the point x. Then we can construct a homotopy by just moving one copy of I^n to the other face. This also shows that it doesn't matter which face we pick. We will leave out the base point from the notation today, but it is always there implicitly.

The most basic non-contractible topological spaces are spheres, and so a question one might ask is what $\pi_n(S^k)$ is. This question ends up being very hard, we don't have a general answer, so we try to calculate specific cases. We would like to calculate $\pi_3(S^2)$ and to do so we need some tools.

Fibre Bundles

A fibre bundle of fibre type F over a topological space B is a topological space E along with a map $\pi : E \to B$ such that for $b \in B$ we have $\pi^{-1}(b) \cong F$ and also a neighbourhood U of b such that there is an isomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\pi_1"] \\
U &
\end{tikzcd}$$

commutes.

One example of a fibre bundle is the tangent bundle, here the fibre type is a vector space, but today we're only interested in viewing this vector space as a topological space.

Another example of a fibre bundle is a covering space $\tilde{X} \to X$ where the fibre is a discrete set S discrete.

A Long Exact of Higher Homotopy Groups

If we have a fibre bundle we can include the base along on of the fibres, then we have the exact sequence $F \to E \to B$ in the category of pointed topological spaces. What that means is that the image of a map, is the kernel of the next. In pointed spaces we consider the kernel to be points mapping to the marked point. This then becomes, and we won't go through the proof, a long exact sequence of higher homotopy groups

$$\begin{tikzcd}
\pi_3(F) \arrow[r] & \pi_3(E) \arrow[r] & \pi_3(B) \arrow[dll, rounded corners] & \\
\pi_2(F) \arrow[r] & \pi_2(E) \arrow[r] & \pi_2(B) \arrow[dll, rounded corners] & \\
\pi_1(F) \arrow[r] & \pi_1(E) \arrow[r] & \pi_1(B) \arrow[dll, rounded corners] & \\
\pi_0(F) \arrow[r] & \pi_0(E) \arrow[r] & \pi_0(B) \arrow[r] & 0
\end{tikzcd}.$$

For our example of coving spaces this becomes

$$\pi_1(S) \longrightarrow \pi_1(\tilde{X}) \longrightarrow \pi_1(X) \longrightarrow \pi_0(S) \longrightarrow \pi_0(X)$$

making a short exact sequence giving us that $\pi_1(S)$ has $|\pi_1(X)/\pi_1(\tilde{X})|$ points.

The Hopf Fibration

First we construct a map $q : \mathbb{C}^2 \backslash \{0\} \to S^2$, to do this we map the map (z, w) to z/w and think about this as a map to $\mathbb{C} \cup \{\infty\} \cong S^2$. Then we have the diagram

$$\begin{tikzcd} \mathbb{C}^2\backslash\{0\} \arrow[d, "q"'] & S^3 \arrow[l, hook'] \arrow[dl, dashed, "h"] \\ S^2 & \end{tikzcd}.$$

We call the composition h the Hopf Fibration.

The Hopf Fibration makes S^3 into a fibre bundle over S^2. To see this, if we have a point in S^2, we can think of this as a point z in $\mathbb{C}$ using the map φ^{-1} so long as it isn't the point ∞ (and if it is ∞ we could just rotate our sphere first). Then we can look at the pre-image of z. Certainly $(z, 1)$ is in the preimage of z under h and then so is $(z, 1)/\,||(z, 1)||$, and this is actually in S^3. The other points in S^1 should all be multiples of this point $\lambda(z, 1)/\,||(z, 1)||$, but these only lie in the sphere for $|\lambda| = 1$. The preimage of a any neighbourhood U of z is then $\lambda(w, 1)/\,||(w, 1)||$ for $\lambda \in S^1$ and $w \in U$, which we can just map to $(\lambda, w) \in S^1 \times U$ to get our local trivialization. This shows that the Hopf Fibration makes S^3 into a fibre bundle of S^2 with fibre type S^1.

Today we are most interested in the Hopf Fibration as a fibre bundle, but it is of great interest in geometry and the map h isn't just continuous, but a Riemannian submersion. Now, if we steographically project from S^3 to $\mathbb{R}^3$ we get that the fibre circles remain circles in S^3, except for the circle through the point of projection, which becomes a line (or a circle through ∞). This covers $\mathbb{R}^3$ with non-overlapping circles, each of which is linked. It's not to hard to see using the trivialization above that latitudes on S^2 become tori in $\mathbb{R}^3$.

Applying the Long Exact Sequence

Now we can put everything together, we have a way to relate higher homotopy groups given a fibre bundle, and we have a fibre bundle in which everything is a sphere! Here we have the following long exact sequence.

$$\begin{tikzcd}
\pi_3(S^1) \arrow[r] & \pi_3(S^3) \arrow[r] & \pi_3(S^2) \arrow[dll, rounded corners] & \\
\pi_2(S^1) \arrow[r] & \pi_2(S^3) \arrow[r] & \pi_2(S^2) \arrow[dll, rounded corners] & \\
\pi_1(S^1) \arrow[r] & \pi_1(S^3) \arrow[r] & \pi_1(S^2) \arrow[dll, rounded corners] & \\
\pi_0(S^1) \arrow[r] & \pi_0(S^3) \arrow[r] & \pi_0(S^2) \arrow[r] & 0
\end{tikzcd}.$$

We aim to fill in as much of this as possible, in order to calculate the $\pi_3(S^2)$ in the top right.

First, we can immediately fill all the π_1's using our knowledge of the fundamental group.

Second we can fill in the π_0's, these are all just 0 because all our spaces only have one path component.

Next we turn our attention to the left hand column where we look to calculate $\pi_n(S^1)$ for $n > 1$. This is defined to be homotopy classes of maps $f : S^n \to S^1$. Here we need to recall a theorem from covering spaces that says if there is a map $f : X \to Y$ and Y has a universal cover $\tilde{Y}$ then the map f lifts to the universal cover of Y if and only if it induces the 0 map $f_* : \pi_1(X) \to \pi_1(Y)$. Here since $\pi_1(S^n) = 0$ we get that f_* is automatically injective so we get a lift

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ S^n \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & S^1 \end{tikzcd}.$$

Because $\mathbb{R}$ is contractible we can take a homotopy of $\tilde{f}$ to a constant map $F : S^n \times I \to \mathbb{R}$ and then post compose with π to get a homotopy of f to a constant map. This shows that $\pi_n(S^1) = 0$ for $n > 1$.

The last big tool we're going to need is the Freudenthal Suspension Theorem. This theorem says that for $i < 2n - 1$ we get that $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$. Here this isomorphism is induced by the suspension map. We won't prove this, but no big tools are used in the proof, it is mostly just explicit reasoning with cells. If one doesn't like using this theorem we can always use Hurwitz theorem to compute $\pi_2(S^2)$, $\pi_3(S^3)$, and $\pi_2(S^3)$, but Freudenthal allows us to avoid appealing to homology. With this theorem in hand we can compute one more group, we see that $\pi_2(S^3) \cong \pi_1(S^2) \cong \mathbb{Z}$.

Now if we look back on our exact sequence we have

$$\begin{tikzcd} 0 \arrow[r] & \pi_3(S^3) \arrow[r] & \pi_3(S^2) \arrow[dll, rounded corners] & \\ 0 \arrow[r] & 0 \arrow[r] & \pi_2(S^2) \arrow[dll, rounded corners] & \\ \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, rounded corners] & \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}.$$

Exactness here allows us to conclude that $\pi_2(S^2) = \mathbb{Z}$. Its worth considering this result for a moment. These maps our classified by their degree, which intuitively is the number of times a generic point of the codomain S^2 is covered by the domain S^2. One can think about the map from S^2 to S^2 wrapping the sphere around itself twice, under the identification of $\pi_2(S^2) = \mathbb{Z}$ this map becomes 2, one shouldn't be too surprised to learn that this map is the suspension of the map of the circle wrapping around itself twice.

Now, another application of Freudenthal allows us to calculate that $\pi_3(S^3) = \pi_2(S^2) = \mathbb{Z}$. These maps our again categorized by degree. Now our sequence becomes

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] & \pi_3(S^2) \arrow[dll, rounded corners] & \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[dll, rounded corners] & \\ \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, rounded corners] & \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}.$$

Finally one last application of exactness allows us to conclude that $\pi_3(S^2) = \mathbb{Z}$, and it shouldn't surprise you that this group is actually generated by the Hopf Fibration.