ASSEMBLY IN THE LAND OF WALDHAUSEN

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These are notes for Alexander Kupers' Simple Homotopy Week. This talk finishes a series discussing Waldhausen's work [Wal85].

1. Introduction

Suppose that we have a functor out of the category of spaces

$$\mathcal{S} \xrightarrow{F} \mathcal{C}.$$

This functor need not be *excisive*. That is, for a space X take the straightening map

$$\operatorname*{colim}_X * \xrightarrow{\simeq} X.$$

This always gives a map

$$F_{\%}(X) \coloneqq \operatorname*{colim}_X F(*) \xrightarrow{\alpha} X,$$

which is usually not an equivalence. This map α is actually a natural transformation of functors

$$F_{\%} \to F$$

called *assembly*. α is called assembly because it is assembled out of these equivalencies for each point in X, that is, α is the natural comparison between F and $F_{\%}$, the "best excisive approximation for F". These assembly maps are uniquely characterized by they do to points.

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2. A-Theoretic Assembly and Wall's Obstruction

In the case of the A-theory functor

$$\mathcal{S} \xrightarrow{A} \mathcal{S}$$

we get an assembly map $A_{\%}(X) \to A(X)$. When X is connected, on π_0 this becomes expected map

$$K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\pi_1(X)]).$$

The cofibre of this map is then $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$, and further its not hard to see that the image of the K-theory class of $[C_\bullet(X)]$ in the fibre $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ is exactly Wall's obstruction $w(X)$. This is our first motivation for the following.

Theorem 2.1. *There is a cofibre sequence*

$$A_{\%}(X) \xrightarrow{\alpha} A(X) \to \mathrm{Wh}(X).$$

Here $\mathrm{Wh}(X)$ *is the Whitehead spectrum of* X*.*

This will allow us to rephrase vanishing problems for anything in the image of the map $A(X) \to \mathrm{Wh}(X)$ to lifting problems along the assembly map. This justifies us deeming $A_{\%}(X)$ to be the moduli space of simple homotopy theory.

Example 2.2. As an example, we can lift Wall's obstruction for a connected space finite space X. In the case of π_0 we have a very useful splitting of the A-theoretic assembly map, given by the map $\pi_X : X \to *$. We will show that $\chi(X) = (\pi_X)_![C_\bullet(X)]$ is a lift of $[C_\bullet(X)]$. This $\chi(X)$ is the euler characteristic of X.

To prove this we will induct on cells of X. This is clearly true for a point. Now suppose that for A, B, and C we know that the euler characteristic lifts the K-theory class of cochains. Suppose we have a pushout $B \sqcup_A C$, then let i_A, i_B, and i_C be the inclusions of these spaces into the pushout. Then we know that

$$C_\bullet(B \sqcup_A C) = (i_B)_! C_\bullet(B) \sqcup_{(i_A)_! C_\bullet(A)} (i_C)_! C_\bullet(C)$$

and so using the additivity theorem we get

$$[C_\bullet(B \sqcup_A C)] = [(i_B)_! C_\bullet(B)] + [(i_C)_! C_\bullet(C)] - [(i_A)_! C_\bullet(A)].$$

Since we also have $\chi(B \sqcup_A C) = \chi(B) + \chi(C) - \chi(A)$ we get that the euler characteristic of the pushout lifts the K-theory class of cochains. Since X is finite this completes the induction.

3. Waldhausen's Model of Assembly

Remark 3.1. To begin, we note that what is to follow is in the language of simplicial sets and the like. There is no completely derived version of this story, because we are interested in studying *simple* homotopy theory. A map being simple is *not* a model independent notion, it depends on the specific finiteness structure a space has. One can provide a model independent notion of a finiteness structure, but that goes through exactly the technology we are about to discuss.

We follow very closely the original presentation of this material given in [Wal85].

In this section X will be a simplicial set. Let $\mathcal{R}(X)$ denote the category of retractive simplicial simplicial sets over X, that is the full subcategory of the usual over category $\mathbf{sSet}_{/X}$ given by retracts, that is the map $r\colon Y \to X$ should have a homotopy right inverse. We denote by $\mathcal{R}_f(X)$ the full subcategory where the space Y is a finite simplicial set. Finally we use the superscript h or s denote the full sub category where the require the maps r to be homotopy equivalencies or simple homotopy equivalencies respectively. As a final notational remark, the category $S_\bullet\mathcal{R}_f(X)$, inherits the notion of equivalence and simple equivalence from these notions for simplicial sets.

Theorem 3.2. *Let* X *be a simplicial set. Then there is a pushout square*

$$\begin{tikzcd} sS_\bullet\mathcal{R}^h_f(X^{\Delta^\bullet}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & hS_\bullet\mathcal{R}^h_f(X^{\Delta^\bullet}) \arrow[d] \\ sS_\bullet\mathcal{R}_f(X^{\Delta^\bullet}) \arrow[r] & hS_\bullet\mathcal{R}_f(X^{\Delta^\bullet}) \end{tikzcd}$$

and applying the geometric realization functor this gives

$$\begin{tikzcd} \mathrm{Wh}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\ \Omega^{-1}A_{\%}(X) \arrow[r, "\Omega^{-1}\alpha"'] & \Omega^{-1}A(X) \end{tikzcd}$$

3.1. **Preliminaries.** To begin, we note that the equivalences
$$\Omega|hS_\bullet\mathcal{R}_f(X^{\Delta^\bullet})| \simeq A(X)$$
and
$$sS_\bullet\mathcal{R}_f^h(X^{\Delta^\bullet}) \simeq \mathrm{Wh}(X)$$
are simply the definitions from the previous talk.

Proposition 3.3.

3.2. **A-Theoretic Homology.**

Proposition 3.4. *The functor $\Omega|sS_\bullet\mathcal{R}_f(X^{\Delta^\bullet})|$ is excessive. Moreover, this functor is a model for the functor $A_\%(X)$.*

Proof Sketch. Here we have a chain of equivalences given by
$$sS_\bullet\mathcal{R}_f(X^{\Delta^\bullet}) \leftarrow sS_\bullet\mathcal{R}_f(X_\bullet) \rightarrow hS_\bullet\mathcal{R}_f(X_\bullet)$$
Here the first map comes from the identification $X_n = (X^{\Delta^n})_0$. The second two terms are just usual models for the homology theories associated to the infinite loop spaces $sS_\bullet\mathcal{R}_f(*)$ and $hS_\bullet\mathcal{R}_f(*)$ respectively. Now if for a moment we assume that the first term is a homology theory then to check that the maps above induce equivalences on a point.

All that remains is to check that the first term is a homology theory, and the only part we actually need to check is excision. The first step is to reduce to showing that the functor $sS_\bullet\mathcal{R}_f(X)$ is excessive, which is purely formal and we won't detail. Now, we can draw the diagram
$$\begin{tikzcd}
sS_\bullet\mathcal{R}_f(X_0) \arrow[r] \arrow[d] & sS_\bullet\mathcal{R}_f(X_1) \arrow[r] \arrow[d] & sS_\bullet\mathcal{R}_f(X_1 \sqcup_{X_1} X_0) \arrow[d] \\
sS_\bullet\mathcal{R}_f(X_2) \arrow[r] & sS_\bullet\mathcal{R}_f(X_1 \sqcup_{X_0} X_2) \arrow[r] & sS_\bullet\mathcal{R}_f(X_1 \sqcup_{X_0} X_2, X_2)q
\end{tikzcd}$$
by taking some reasonable definition of $\mathcal{R}_f(X, A)$ one gets this as the cofibre using the additivity theorem, and also that the right most vertical map is an isomorphism. □

A very good question is why this functor is excessive but the analoge where s is h (i.e. the one which gives $A(X)$) is not. The idea is that being a simple equivalence is a *pointwise property* it is saying something about the shape of the fibre. Being a homotopy equivalence on the other hand, is not a pointwise property.

Corollary 3.5. *The map*
$$sS_\bullet\mathcal{R}_f(X^{\Delta^\bullet}) \rightarrow hS_\bullet\mathcal{R}_f(X^{\Delta^\bullet})$$
induces the assembly map.

Proof. It suffices to check this on a a contractible space in which case it is clear. □

3.3. **Proving the Main Theorem.** The big piece of input to actually prove this will be the Fibration Theorem, proved in an earlier talk.

Theorem 3.6 (Fibration Theorem)**.** *If $\mathcal{C}$ is a category with a cylinder functor that is sufficiently nice, with two notions of weak equivalence $\simeq_w$ and $\simeq_v$ one of which refines the other then we get a pullback square*
$$\begin{tikzcd}
vS_\bullet\mathcal{C}^w \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & wS_\bullet\mathcal{C}^w \arrow[d] \\
vS_\bullet\mathcal{C} \arrow[r] & wS_\bullet\mathcal{C}
\end{tikzcd}.$$
Further, the top left is contractible.

Proof of Theorem 3.2. First we note that the contractibility assumption gives us the last identification we needed.

Now, using a technical lemma about bisimplicial sets, it is enough to show that we have a pushout square for every n

$$\begin{tikzcd} sS_\bullet\mathcal{R}_f^h(X^{\Delta^n}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & hS_\bullet\mathcal{R}_f^h(X^{\Delta^n}) \arrow[d] \\ sS_\bullet\mathcal{R}_f(X^{\Delta^n}) \arrow[r] & hS_\bullet\mathcal{R}_f^h(X^{\Delta^n}) \end{tikzcd}.$$

Since X was already arbitrary we can replace X^{Δ^n} with just X. Then this square will come from applying Theorem 3.6 to the functor $\mathcal{R}_f$.

The only technical issue is that the equivalences on the category $\mathcal{R}_f$ are not closed under extensions. To remedy this we replace $\mathcal{R}_f$ with $\mathcal{R}'_f$, the full sub category wherein the maps $r : Y \to X$ are 1-connected. Now the axiom of extension is satisfied, and we just have to verify that passing to $\mathcal{R}'_f$ didn't change anything. This is because applying Σ^2 to the square above is homotopic to the identity by a previous talk, but also to takes $\mathcal{R}_f(-)$ into $\mathcal{R}'_f(-)$.

$\square$

References

[Wal85] F. Waldhausen, *Algebraic k-theory of spaces*, in A. Ranicki, N. Levitt, and F. Quinn, editors, *Algebraic and Geometric Topology*, Springer Berlin Heidelberg, Berlin, Heidelberg, 1985, pp. 318–419. 1, 2